Optimal Stopping Problems: Mathematical Formulation and Solution Techniques

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ABSTRACT

This paper is concerned with the formulation and method of solution of the optimal stopping problem, the existence and uniqueness of the optimal topping time are also present.

KEYWORDS: Stopping time, optimal stopping problem, Variational inequality.

INTRODUCTION

The theory of optimal stopping usually concerned with the problem that choosing a specific time to take a particular action, theses type of problems known as optimal stopping problems ,see[1]. An optimal stopping problems have well known applications in stochastic analysis, control theory and finance. The financial applications of the optimal stopping problem is used to maximize the expected reward function or minimize an expected cost over all stopping times. An example of the applications of optimal stopping problem in finance is the model of the stock price that described by geometric Brownian motion, see [2].

Throughout this paper we will consider the following stochastic setting:

The probability space (Ω, f, p), increasing sequence of sigma algebra $\{F_t\}_{t\in T}$ (the filtration) which satisfy the usual conditions and hypotheses where T is the time interval needed for the experiment, stochastic process X_t which adapted to the filtration $\{F_t\}_{t\in T}$, the stopping time τ and the optimal stopping time τ^* ,see [3].

Definition: A geometric Brownian motion is a continuous-time stochastic process , and we say that the stochastic process S_t fellow the geometric Brownian motion if it satisfy the following stochastic differential equation :

$$d S_t = \mu S_t dt + \sigma S_t dW_t \qquad (1)$$

where μ represents the constant drift or trend of the process , σ represents the amount of random variation around the trend and W_t is a Wiener process or Brownian motion. If we denote by $S_0 = S(0)$, to the initial value which is F_0 $-$measurable, then by using Ito lemma the geometric Brownian motion (1) has general solution of the form:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \qquad (2)$$

FORMULATION OF THE OPTIMAL STOPPING PROBLEM

For the mathematical formulation of the problem of this type we will consider the filtered probability space $(\Omega, f, \{F_t\}_{t\in T}, p)$, and the standard Brownian motion $B=\{B_t \; ; t \geq 0\}$, suppose that the state of the system of this type is described by Geometric Brownian motion , and let g be a given Boreal function (the reward function) on R^n which must satisfy the following condition:

$a)$ $g(\xi) \geq 0 , \forall \; \xi \in R^n$ $\qquad$ $b)$ g is continuous .

Denote by $F = \{ f_t \}$ to the natural filtration of X_t and let $\rho \geq$ be a real constant such that $E\left[e^{-\rho\tau} g(X_\tau)\right]$ exist and well define for every *F*- stopping time τ . Define the value function $\varphi(\tau^*, X)$ of the optimal stopping problem with reward function g and discount rate ρ as:

$$\varphi\left(\tau^{*},X\right)=E^{x}\left[e^{-\rho\tau^{*}}X_{\tau^{*}}\right]=\sup_{\tau}E^{x}\left[e^{-\rho\tau^{*}}g(X_{\tau})\right]_{x\in R^{n}} \qquad (3)$$

Where the supremum is taken over all stopping times τ and E^{x} denotes the expectation with respect to the probability law of the process X_t , $t\geq 0$.

Then the optimal stopping problem is to find the value function $\varphi\left(\tau^{*}, X\right)$ as well as an optimal stopping time τ^{*} for which the supremum is attained if such time exist .

Definition :A measurable function $f:R^{n}\rightarrow(0,\infty]$ is called supermeanvalued with respect to X_t if for all stopping times τ and $\forall x\in R^{n}$.

$$f(x)\geq E^{x}[\ f(X_{t})] \qquad (4)$$

If $f(x)$is also lower semiconscious (l.s.c) , then f is called l.s.c super harmonic. and if $f:R^{n}\rightarrow(0,\infty]$ is l.s.c , then by using Fatou lemma for any sequence $\{\tau_k\}$of stopping times such that $\tau_k\rightarrow 0$ a.s we get

$$f(x)\leq E^{x}\left[\lim_{k\to\infty}f\left(X_{t_k}\right)\right]\leq\lim_{k\to\infty}E^{x}\left[f\left(X_{t_k}\right)\right] \qquad (5)$$

By combining (4) and (5) we see that for such sequence τ_k, if $f(x)$ is l.s.c super harmonic then,

$$f(x)=\lim_{k\to\infty}E^{x}\left[\ f\left(X_{\tau_k}\right)\right]\quad ,\ \forall x \qquad (6)$$

Denote by A to the characteristic operator of X_t. If $f\in C^{2}\left(R^{n}\right)$then by Dynkin's formula $f(x)$ is super harmonic w .r . t X_t iff

$$A f\leq 0 \qquad (7)$$

Definition : Let h be a real measurable function on R^{n}, if $f(x)$ is super harmonic (super mean valued) function and $f(x)\geq h$ we say that $f(x)$ is super harmonic (super mean valued) majorant of h (w.r. to X_t) . The function $\bar{h}(x)$ such that

$$\bar{h}(x)=\inf_{f}\ f(x)\ \ ;x\in R^{n} \qquad (8)$$

Where the inf being taken over all super mean valued majorants of h , is called the least super mean valued majorant of h . Suppose now there exists a function $\overset{\frown}{h}$ such that $\overset{\frown}{h}$ is super harmonic majorants of h and If f is any other super harmonic majorants of h then $\overset{\frown}{h}\leq f$ then $\overset{\frown}{h}$ is called the least superharmonic majorants of h.

Let $g\geq 0$ and let $f(x)$ be a supermeanvalued majorant of g. then if τ is an stopping time

$$f(x)\geq E^{x}\left[f\left(X_{\tau}\right)\right]\geq E^{x}\left[g\left(X_{\tau}\right)\right]\quad ,\quad f(x)\geq\sup_{\tau}E^{x}\left[g\left(X_{\tau}\right)\right]=g^{*}(x).$$

Then we have

$$\overset{\frown}{g}(x)\ \geq\ g^{*}(x)\quad ,\quad \forall\ x\in R^{n} \qquad (9)$$

Definition :A lower semi continuous function $f:R^{n}\rightarrow(0,\infty]$ is called excessive (w.r.t X_t) if

$$f(x)\geq E^{x}[f(X_{s})]\quad ,\ \forall, s\geq 0,\ x\in R^{n} \qquad (10)$$

Theorem : (Existence for optimal stopping) see [1]

Let g^{*} denote the optimal reward and $\overset{\frown}{g}$ the least super harmonic majorants of a continuous reward function $g\geq 0$, then

$$i)\quad g^{*}(x)=\overset{\frown}{g}(x) \qquad (11)$$

$ii)$ for $\varepsilon>0$ define the domain D as:

$$D_{\varepsilon} = \left\{ x \; ; \; g(x) < \overset{\frown}{g}(x) - \varepsilon \right\} \tag{12}$$

Suppose that the function g is bounded , then the optimal time τ_{ε} is the first time that the process exit from the domain .the domain . D_{ε} is close to being optimal ,in the sense that :

$$\left| \, g^{*}(x) - E^{x}\left[\, g\left(X_{\tau_{\varepsilon}} \right) \right] \right| \le 2\varepsilon \quad , \forall \; x \tag{13}$$

iii) For arbitrary continuous $g \ge 0$ Let

iv) $$D = \left\{ x \; ; \; g(x) < g^{*}(x) \right\} \tag{14}$$

be the continuation region

For $N = 1, 2, 3$ define $g_N = g \wedge N$, $D_N = \{ \, x \, ; g_N(x) < \hat{g}_N(x) \, \}$ and $\sigma_N = \tau_{N_D}$. Then $D_N \subset D_{N+1}$, $D_N \subset D \cap g^{-1}(0, N)$, $D = \bigcup_N D_N$, If $\sigma_N < \infty$ a.s. Q^x for all N then

$$g^{*}(x) = \lim_{N \to \infty} E^{x}\left[g\left(X_{\sigma N} \right) \right] \tag{15}$$

v) In particular ,if $\tau_D < \infty$ a.s . Q^x and the family $\left\{ g\left(X_{\sigma N} \right) \right\}_N$ is

vi) uniformly integrable w.r.t . Q^x .

Let the value function g^{*} (the stopping function) be

$$g^{*}(x) = E^{x}\left[\, g\left(X_{\tau_D} \right) \right] \tag{16}$$

And $\tau^{*} = \tau_D$ is the optimal stopping time.

Theorem: (Uniqueness for optimal stopping) see [1]

Define the domain D such that:

$$D = \left\{ x \; ; \; g(x) < g^{*}(x) \right\} \subset R^{n} \tag{17}$$

Suppose there is exist an optimal stopping time $\tau^{*} = \tau^{*}(x, \omega)$ for the

problem (3) for all x. then

$$\tau^{*} \ge \tau_D \quad , \quad \textit{for all} \quad x \in D \tag{18}$$

and

$$g^{*}(x) = E^{x}\left[\, g\left(X_{\tau_D} \right) \right] \quad , \quad \textit{for all} \;\; x \in R^{n} \tag{19}$$

Hence τ_D is an optimal stopping time for the problem (3), for the proof see [1]

Theorem : (Variational inequality for optimal stopping) see [1]

Suppose that we can find a function $\phi : \overline{V} \to R$ such that the following conditions are hold

$(i)\ \ \phi \in C(V) \cap C(\overline{V})$

$(ii)\ \ \phi \geq g \ \ on\ V \ \ and \ \ \phi = g \ \ on\ \partial V$

Define the domain $D = \{x \in V\ ;\ \varphi(x) > g(x)\}$. and suppose Y_t spend zero time on $\partial V\ a.s\ ,\ i.e$

$(iii)\ E^y\left[\int_0^T \chi_{\partial D}(Y_t)\,dt\right] = 0\ ,\ \ \forall\ y \in V$

And suppose that

(iv) ∂D is a Lipschitz surface , i.e ∂D is locally graph of a function

$h : R^{n-1} \to R$ such that there exist $k < \infty$ with

$$|h(x) - h(y)| \leq K|x - y| \quad , \quad \forall x, y$$

Also suppose that:

(v) $\phi \in C^2(V \setminus \partial D)$ and the second order derivatives of ϕ are locally bounded near ∂D

$(vi)\ \ L\phi + f \leq 0 \ \ on\ V \setminus \partial D$

$(vii)\ \ L\phi + f = 0 \ \ on\ \ D$

$(viii)\ \ \tau_D = \inf\{t > o\ ;\ y_t \notin D\} < \infty\ ,\ a.s.\ \ R^y\ \ \forall\ y \in V$ and

(ix) the family is uniformly integrable w.r.t. R^y , $\forall\ y \in V$, then

$$\phi(y) = \Phi(y) = \sup_{\tau \leq T} E^y\left[\int_0^\tau f(y_t)\,dt + g(y_\tau)\right];\qquad y \in V \tag{20}$$

And

$$\tau^* = \tau_D \tag{21}$$

Which is an optimal stopping time for this problem .

METHOD OF SOLUTION OF THE OPTIMAL STOPPING PROBLEM

We can solve the optimal stopping problem by using the variational inequality theorem which state that if we can find a function ϕ which satisfy the conditions in the above theorem , then we can find a solution $\phi = \Phi$ and time τ^* which is an optimal stopping time for the problem.

Application: Now we want give an example to illustrate the formulation of an optimal stopping problem in selling assets.

For the formulation we will consider the following assumptions

- X_t is the price of a person's assets at time t on the open market which described by Geometric Brownian motion .
- There is fixed sale tax or transaction cost $a > 0$ and discounting factor $\rho > 0$
- The discounted net of the sale is given by $e^{-\rho t}(X_t - a)$

In this case the optimal stopping problem is to find the stopping time τ which maximizes $E^{(sx)}\left[e^{-\rho\tau}(X_\tau - a)\right]$ (maximizes the expected profit), if so then τ will be the right time to sell the stocks, see [1].

REFERENCE

1. Oksendal, Bernt K. Stochastic Differential Equations: An Introduction with Applications, Springer, (2002), ISBN 3-540-63720-6.
2. Zhijun Yang . Geometric Brownian motion Model in Financial Market.
3. Thomas G. Kurtz: Lectures on Random Analysis. University of Wisconsin- Madison , WI 53706-1388 September (2001).
4. Hull, John .Options, Futures, and other Derivatives (2009), (7 ed.).